The coupling strikes back

Exploring positivity properties of exact observables in $\mathcal{N} = 4$ super Yang-Mills

Based on M.Sc. thesis supervised by Henn, Korchemsky, Raman

Maximilian Haensch, 06.22.2026

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This concerns the kinematical dependence

What about the coupling?

Warm-up example: Quantum Mechanics

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Ground state energy

$$E_0(\lambda)$$

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Could there be an analogues positivity property in quantum field theory?

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These functions are also known as Stieltjes functions

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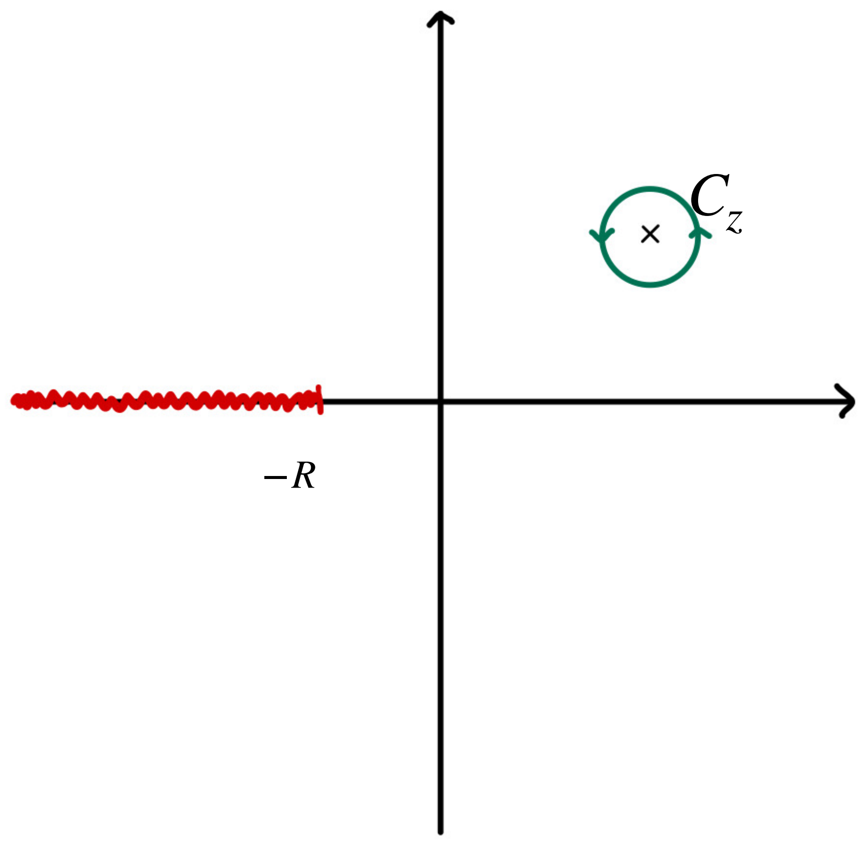

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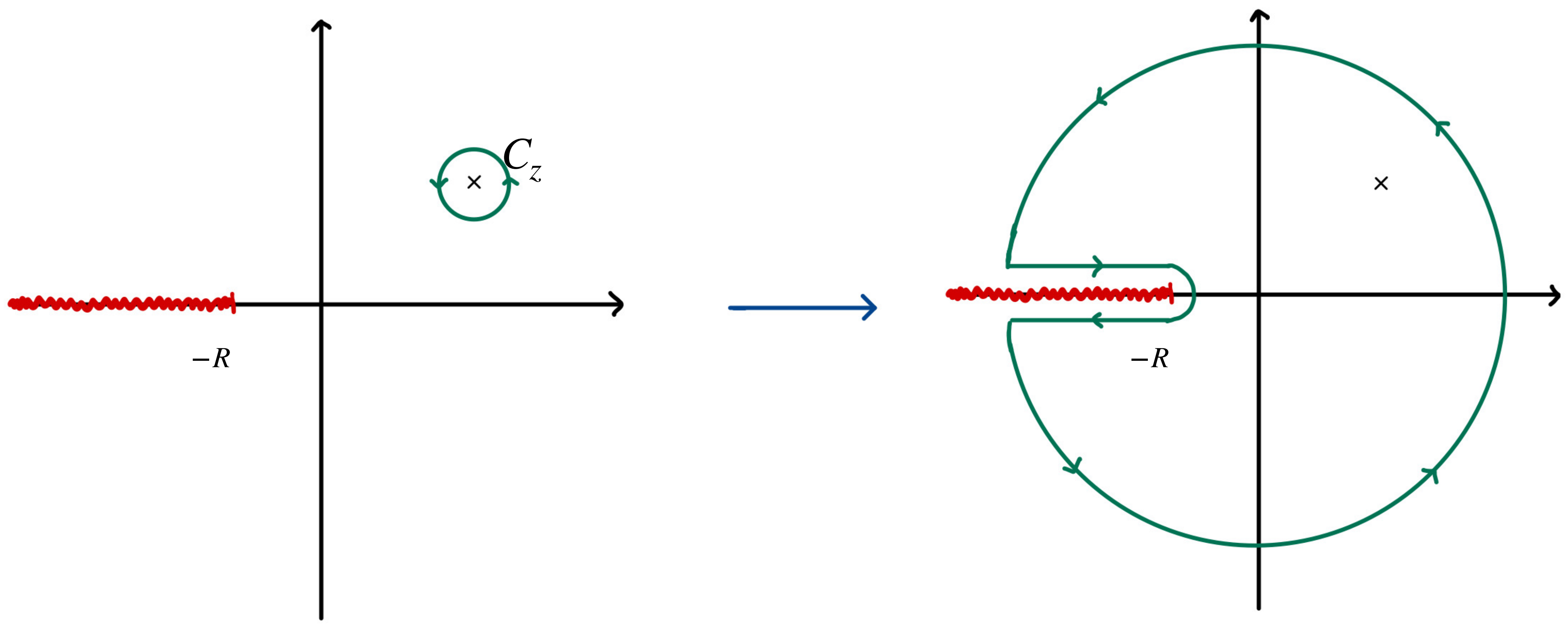

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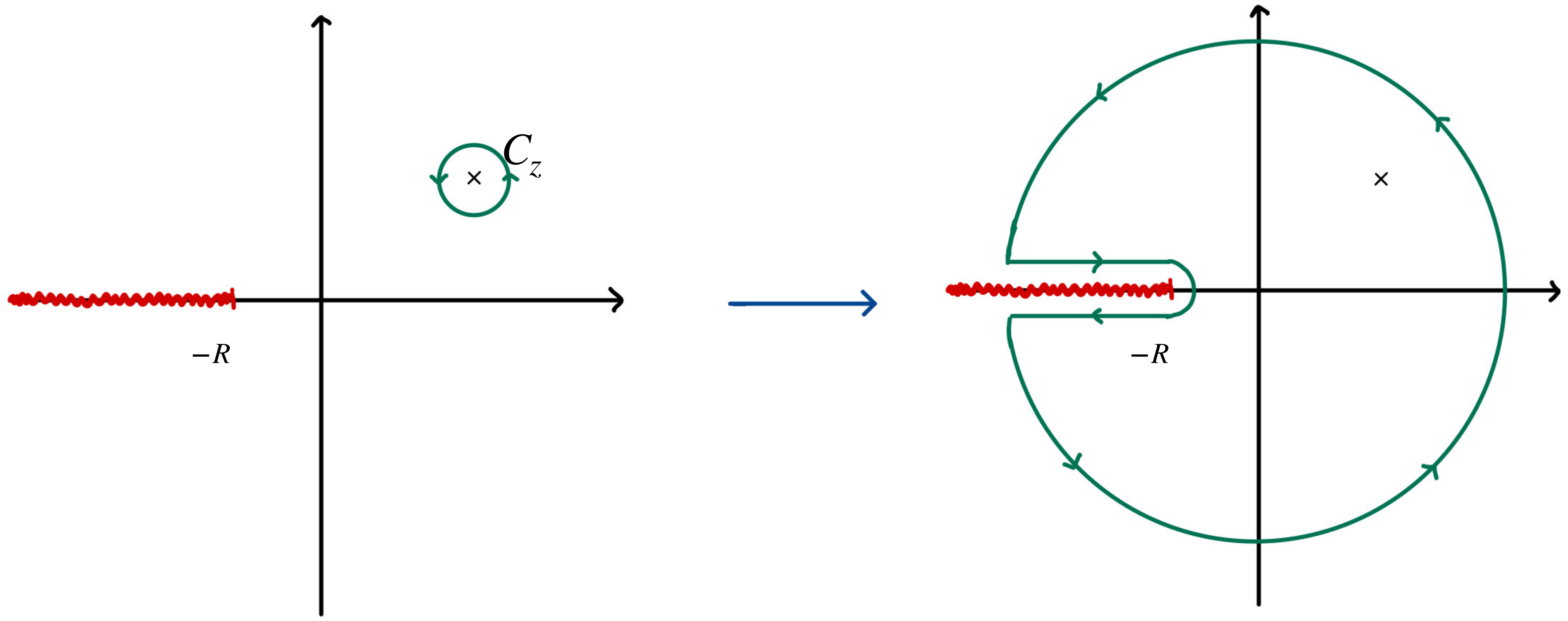

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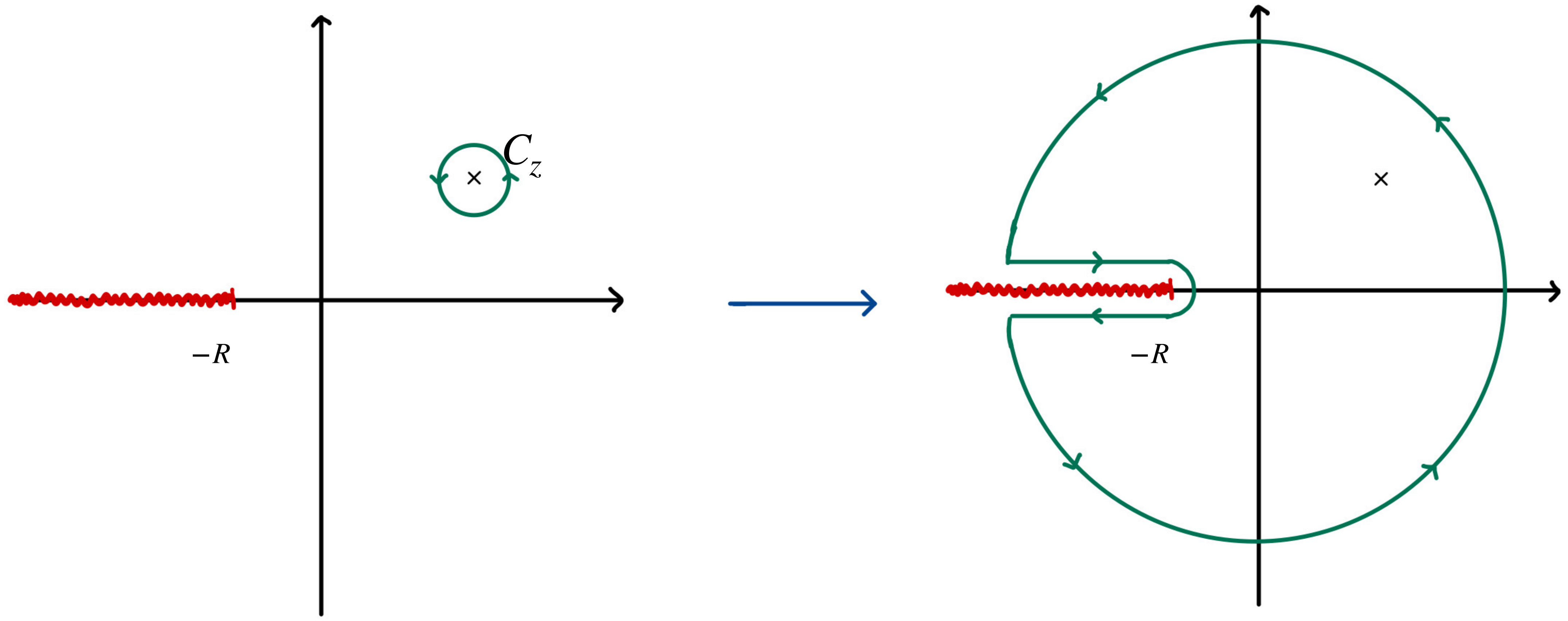

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Hankel matrices are positive semidefinite

$\iff$ all eigenvalues are non-negative

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Example: $N = 3$

$$H_0^3 = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix} \geq 0$$

$$\boxed{\begin{array}{lll} a_1 \geq 0, & a_1 a_3 \geq a_2^2, & \\ a_2 \geq 0, & \dfrac{a_1}{R} \geq a_2, & \dfrac{a_2}{R} \geq a_3 \end{array}}$$

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We are interested in once-subtracted dispersion relations

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We are interested in once-subtracted dispersion relations

$$\frac{\mathcal{O}(\lambda)}{\lambda} = \int_R^{\infty} dt \, \frac{1}{\lambda + t}\mu(t)$$

Slight abuse of terminology:

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Laboratory for studying the coupling dependence of observables

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$\longrightarrow$

Laboratory for studying the coupling dependence of observables

Throughout: $\lambda = g_{\mathrm{YM}}^2 N$ 't Hooft coupling

$g^2 = \frac{\lambda}{(4\pi)^2}$ planar Yang-Mills coupling

Exactly Known Observables

Anomalous dimensions

Correlation functions

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Cusp Anomalous Dimension Γ_{cusp} [Beisert, Eden, Staudacher; 2007]

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Correlation functions

- Governs UV divergence of cusped light-like Wilson loop
  [Polyakov; 1980]
- Controls large spin limit of twist-two anomalous dimension
  [Korchemsky; 1989]
- Controls soft/collinear divergences of amplitudes
  [Korchemsky, Radyushkin; 1987]

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$$\Gamma_{\text{cusp}} = 4g^2 \left(\frac{1}{1+K} \right)_{11}$$

[Benna, Benvenuti, Klebanov, Scardicchio; 2007]

Exactly Known Observables

Anomalous dimensions

Cusp Anomalous Dimension Γ_{cusp} [Beisert, Eden, Staudacher; 2007]

Correlation functions

Can be computed at finite coupling thanks to integrability

$$K_{ij} = 2j(-1)^{ij+1} \int_0^\infty \frac{dt}{t} \frac{J_i(2gt) J_j(2gt)}{e^t - 1}$$

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$$\Gamma_{\text{cusp}} = 4g^2 - \frac{4}{3}\pi^2 g^4 + \frac{44}{45}\pi^4 g^6 - \dots \qquad \text{for } g \to 0$$

$$\Gamma_{\text{cusp}} \sim 2g - \frac{3\log 2}{2\pi} + \dots \qquad \text{for } g \to \infty$$

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$$\Gamma_{\text{cusp}} = \frac{1}{4\pi^2}\lambda - \frac{1}{192\pi^2}\lambda^2 + \frac{11}{46080\pi^2}\lambda^3 - \ldots \quad \text{for } \lambda \to 0$$

$$\Gamma_{\text{cusp}} \sim \frac{1}{2\pi}\sqrt{\lambda} - \frac{3\log 2}{2\pi} + \ldots \quad \text{for } \lambda \to \infty$$

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Appears in:

- „Origin" limits of 6 gluon MHV amplitudes [Basso, Dixon, Papathanasiou; 2020]
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Can be computed exactly!

$$\Gamma_{\text{oct}}(g) = \log\cosh(2\pi g)$$

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- Controls small angle expansion of angle-dependent cusp anomalous dimension
- Related to the circular Wilson loop, which can be computed using localization [Pestun; 2012]

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- Γ_α is a one-parameter generalization of Γ_{cusp}, $\alpha \in [0,\pi/2]$
- $\Gamma(\alpha = 0) = \Gamma_{\text{oct}}$, $\Gamma(\alpha = \pi/4) = \Gamma_{\text{cusp}}$, $\Gamma(\alpha = \pi/3) = \Gamma_{\text{hex}}$
- Appears in „origin" limits of 6 Gluon MHV amplitudes
- Other values of α appear for higher point amplitudes
- Finite coupling result similar to Γ_{cusp}

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- Expectation value of 1/2-BPS operators with $\Delta \to \infty$
- Yields Γ_{oct} in null expansion [Belitsky, Korchemsky; 2020]

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_1 \mathcal{O}_3 \rangle = \frac{\mathbb{O}(u, v)^2}{(x_{12}^2 x_{34}^2 x_{13}^2 x_{24}^2)^{\Delta/2}}$$

- Admits a representation similar to Γ_{cusp}

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Let $\mathcal{O}(\lambda)$ be one of the observables

Weak coupling: $\mathcal{O}(\lambda) = a_1\lambda - a_2\lambda^2 + a_3\lambda^3 - \dots$

finite radius of convergence R

Strong coupling: $\mathcal{O}(\lambda) \sim b\sqrt{\lambda}$ for $\lambda \to \infty$

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Could it be that?

$$\frac{\mathcal{O}(\lambda)}{\lambda} = \int_R^\infty dt\, \frac{1}{\lambda + t}\,\mu(t), \qquad \mu(t) \geq 0$$

Analytic proofs of Stieltjes property

Want an integral representation:

$$\mathcal{O}(\lambda) = \int_0^{R^{-1}} dt \, \frac{\lambda}{1+\lambda t} \nu(t), \qquad \nu(t) > 0$$

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Weierstrass factorization:

$$\cosh\frac{\sqrt{\lambda}}{2} = \prod_{n=0}^{\infty}\left(1+\frac{\lambda}{4\pi^2(n+1/2)^2}\right)$$

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$$\log(1+A\lambda) = \int_0^{A} dt\, \frac{\lambda}{1+\lambda t}$$

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$R = \pi^2$ radius of convergence

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$$\log W_{\text{circ}}(\lambda) = \log\frac{2I_1(\sqrt{\lambda})}{\sqrt{\lambda}} = \sum_{n=1}^{\infty} \log\left(1+\frac{\lambda}{j_{1,k}^2}\right)$$

$j_{1,k} =$ k th root of $J_1(x)$

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3. Bremsstrahlung

Analytic proofs of Stieltjes property

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Stieltjes property also holds at finite N

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Recall: $\mathscr{O}(\lambda)$ is Stieltjes with $R > 0 \iff \mathscr{O}(\lambda) = \sum_{n=1}^{\infty} (-1)^{n+1} \lambda^n a_n$ with $a_n = \int_0^{1/R} dt\, t^{n-1} \nu(t), \quad \nu(t) \geq 0$

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Hankel matrix constraints:

$$\boxed{H_M^N = \{a_{1+i+j+M}\}_{i,j=0}^{\lfloor (N-M)/2 \rfloor}} \qquad H_0^N \succeq 0 \qquad H_1^N \succeq 0$$

$$\frac{1}{R} H_0^{N-1} - H_1^N \succeq 0 \qquad \frac{1}{R} H_1^{N-1} - H_2^N \succeq 0, \quad \forall N > 0$$

[Bellazzini, Miró, Rattazzi, Riembau, Riva; 2020]

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Tilted cusp anomalous dimension

Constraints verified
up to 20 loops for all
$\alpha \in [0,\pi/2]$

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- In the null limit, $\log \mathbb{O}$ reduces to the octagon anomalous dimension, which is Stieltjes

- Away from this limit, Hankel constraints seem to be violated (work in progress)

Summary

Anomalous dimensions

Cusp Anomalous Dimension Γ_{cusp}

Octagon Anomalous dimension Γ_{oct}

Bremsstrahlung function B

Tilted Cusp Anomalous Dimension Γ_α

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$$P_N^N(\lambda) \leq \mathcal{O}(\lambda) \leq P_N^{N+1}(\lambda)$$

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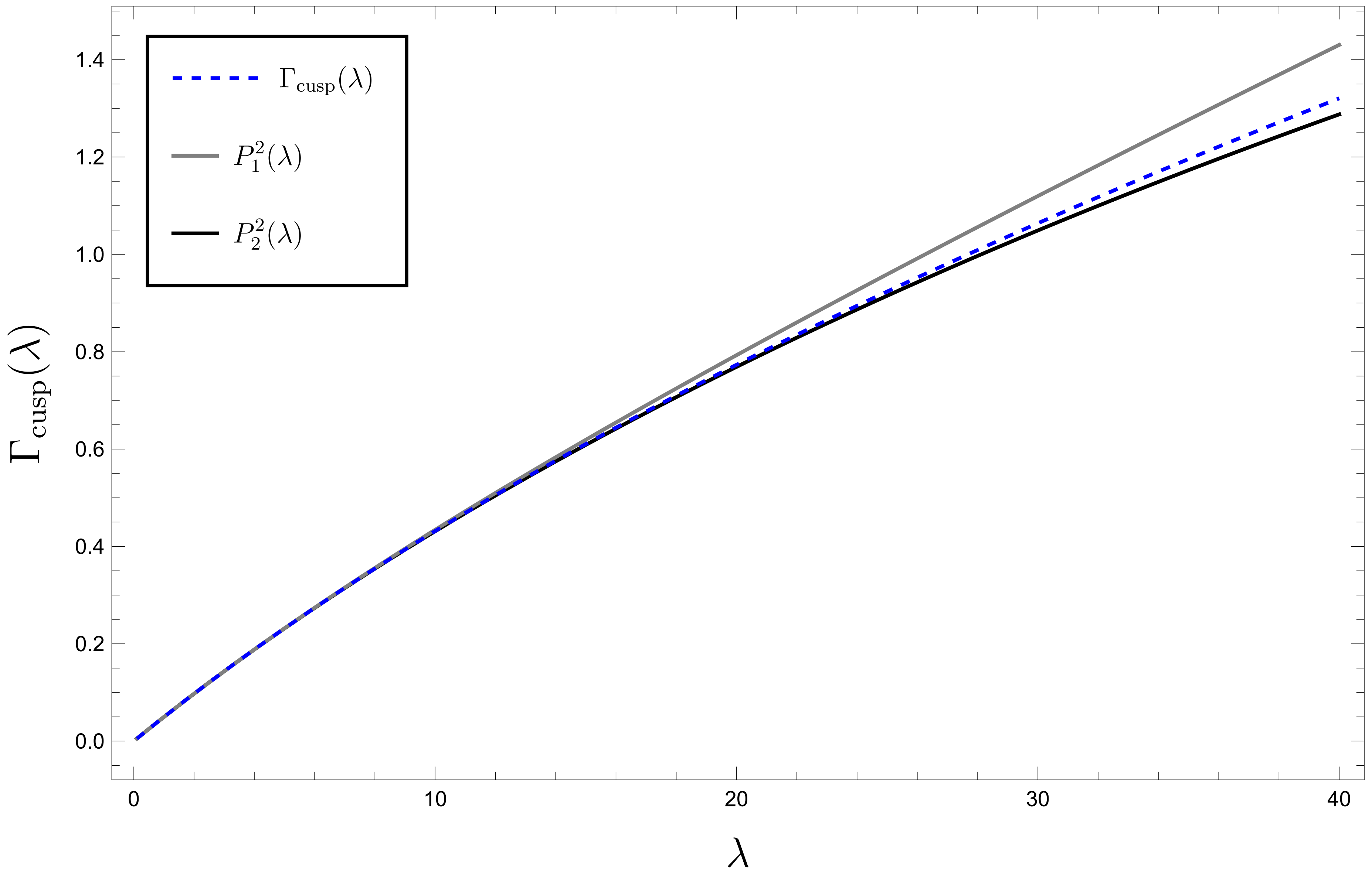

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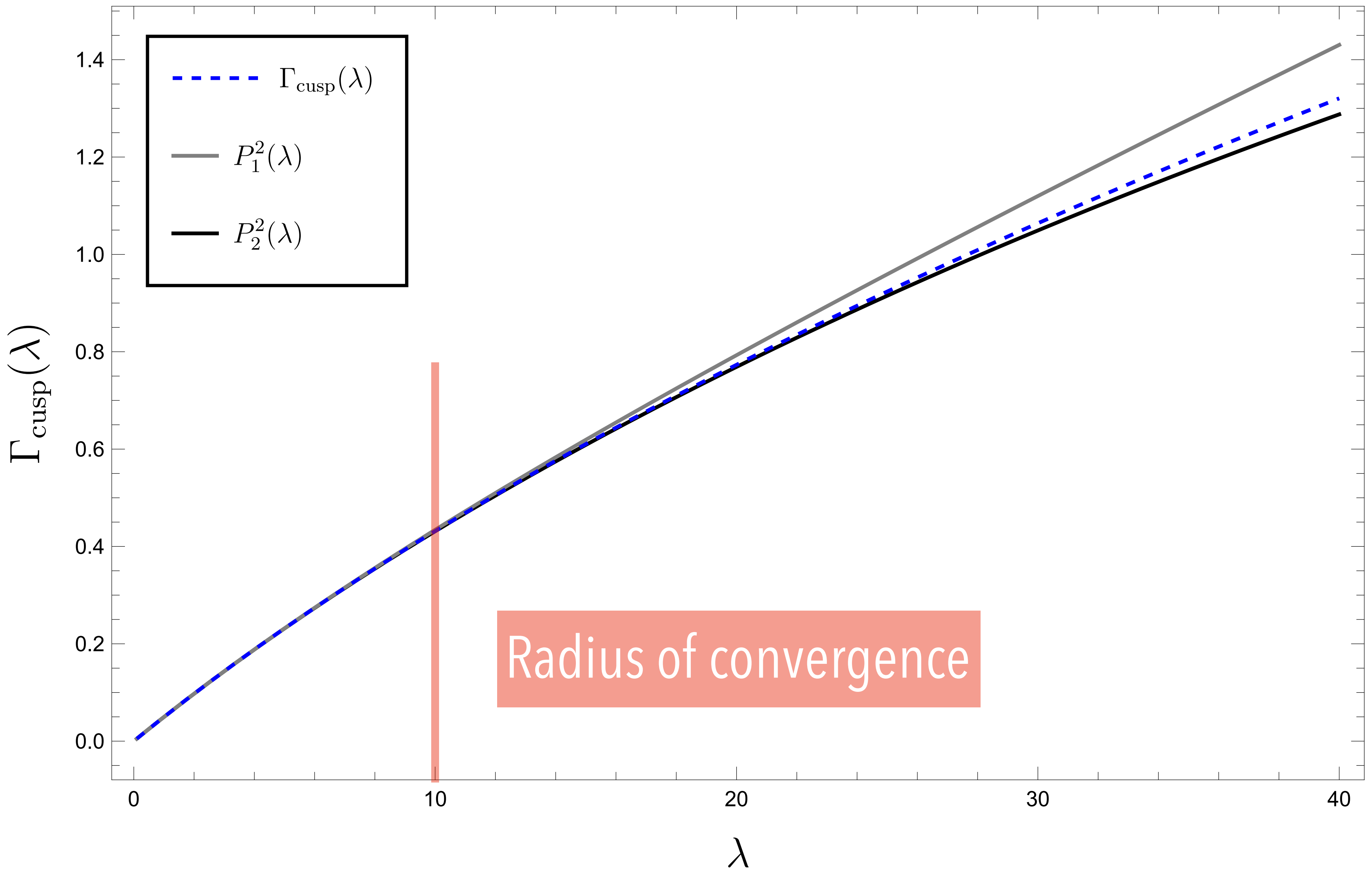

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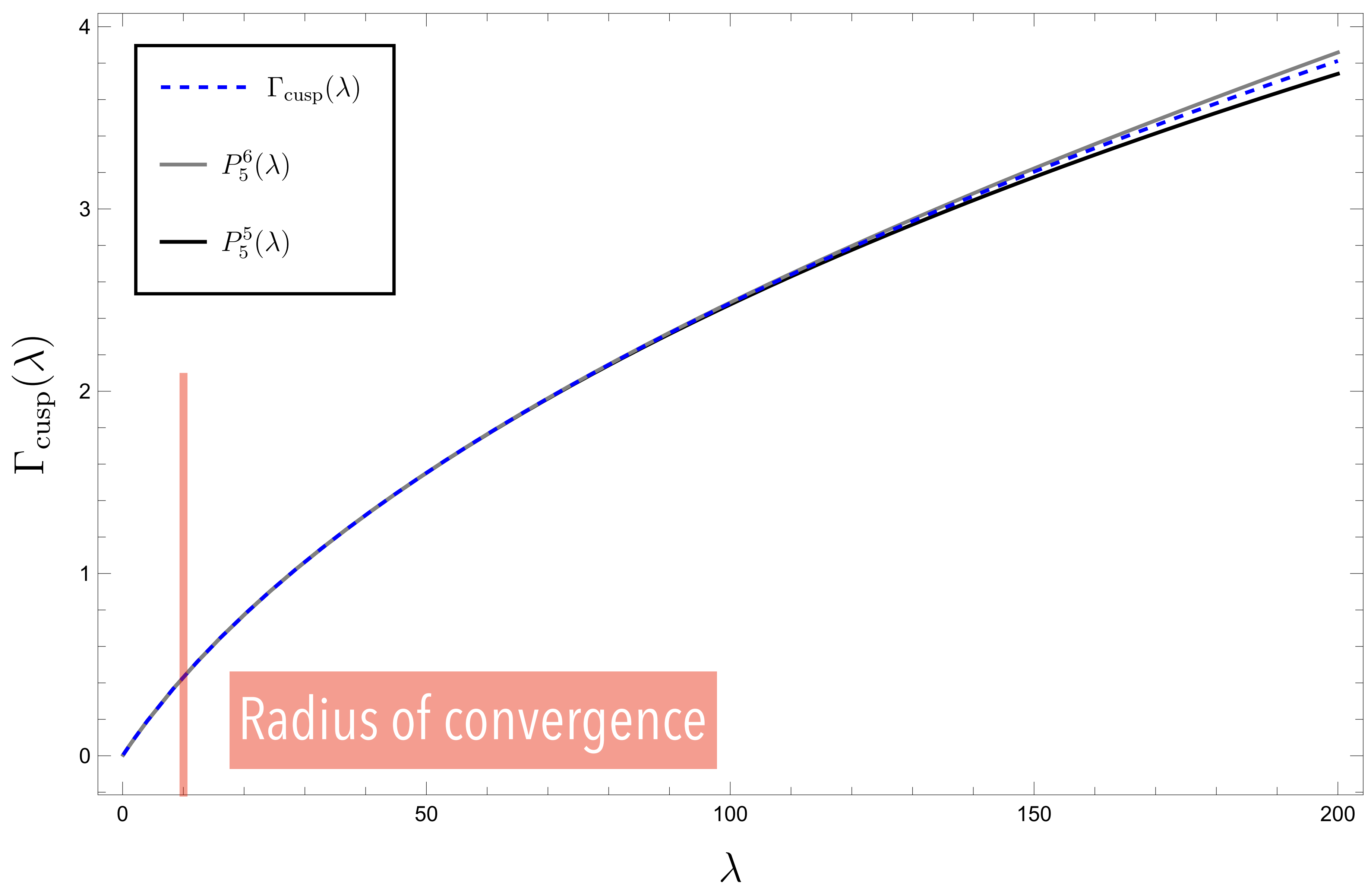

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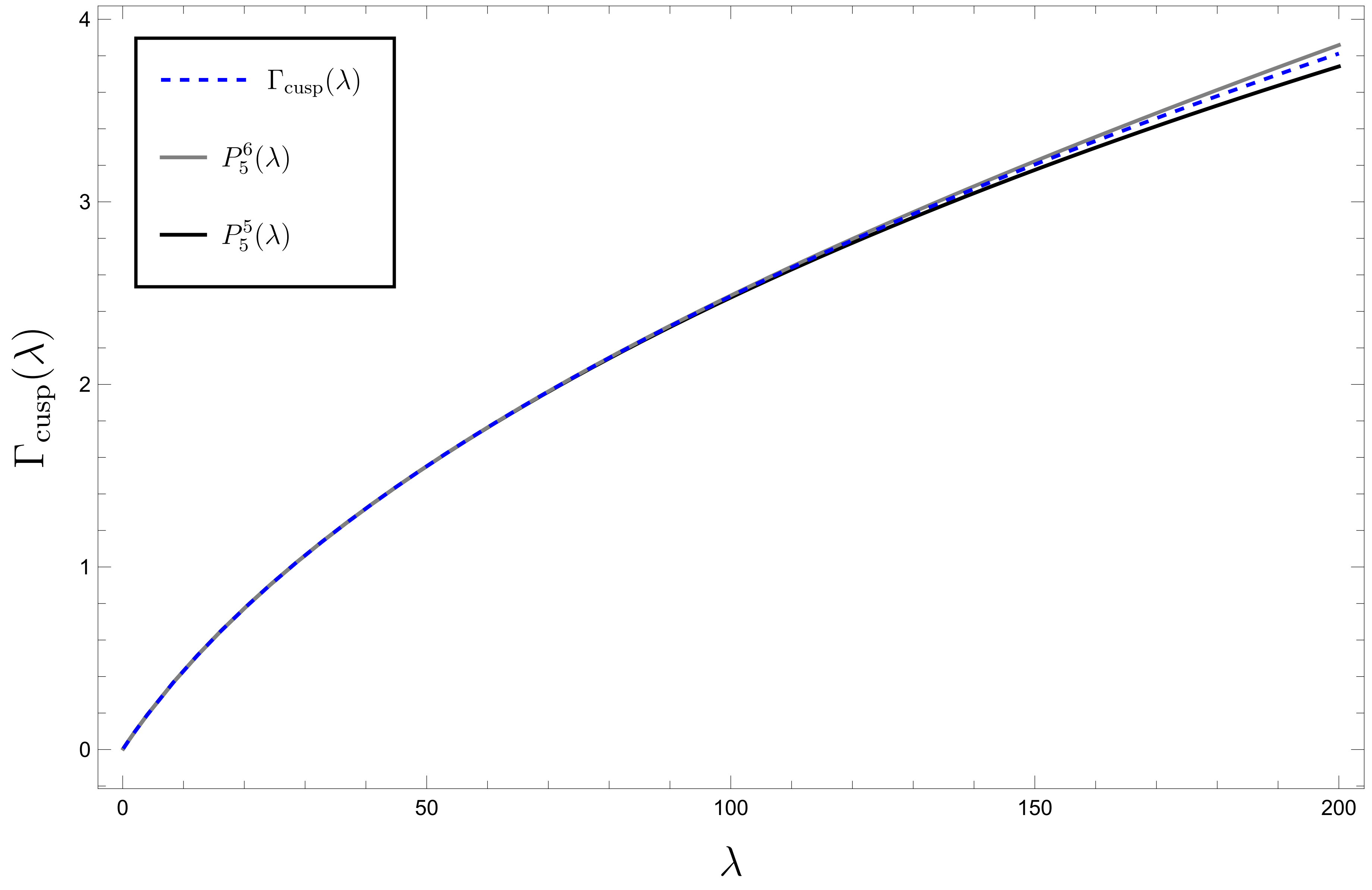

2. Moment Problem Bootstrap

Positivity $\Longrightarrow$ constraints on higher coefficients

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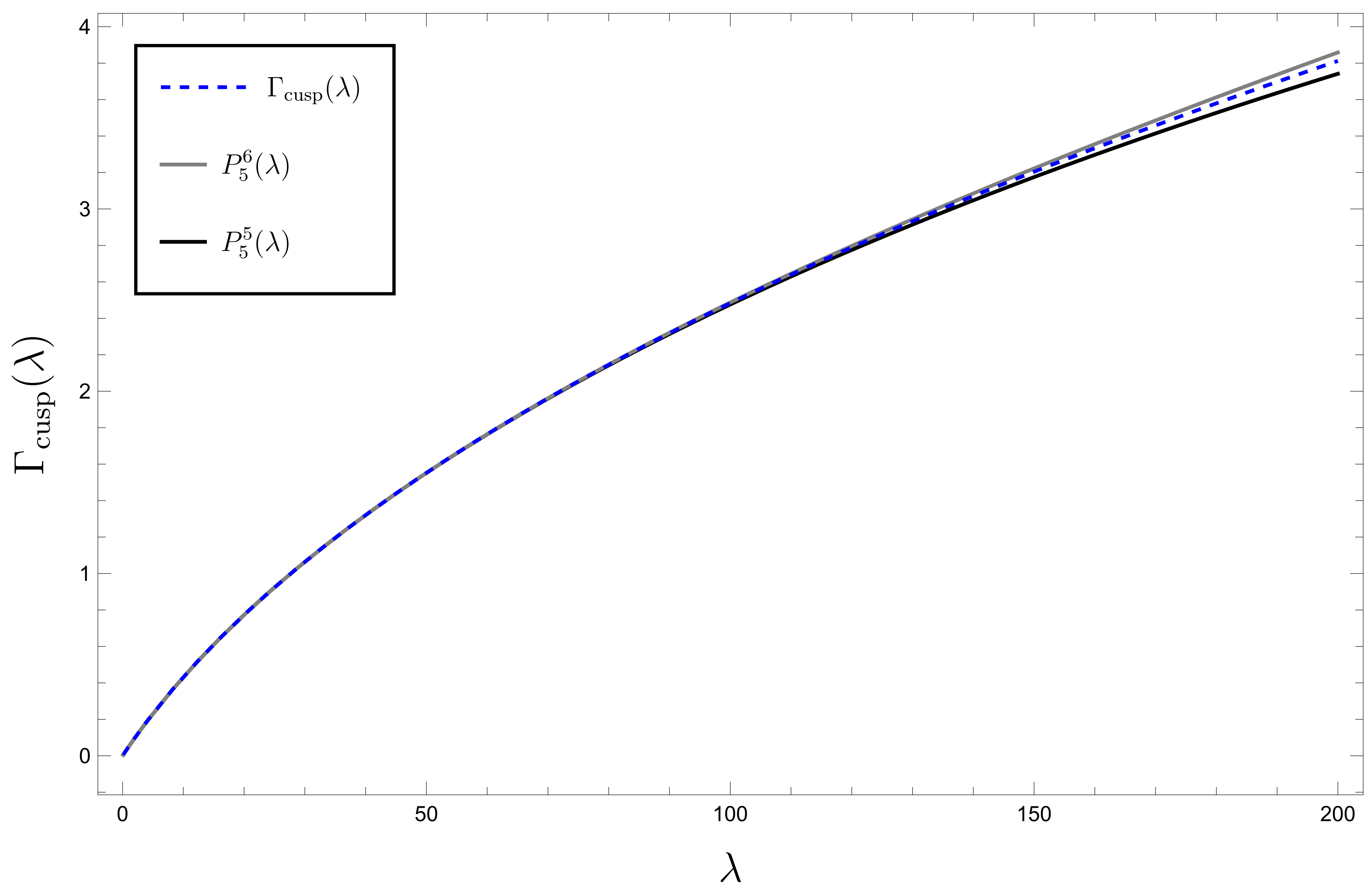

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Allowed region

$$3.21008 \times 10^{-15} < a_{12} < 3.21021 \times 10^{-15}$$

$$2.88697 \times 10^{-16} < a_{13} < 2.8877 \times 10^{-16}$$

$$2.61879 \times 10^{-17} < a_{14} < 2.62108 \times 10^{-17}$$

$$2.39276 \times 10^{-18} < a_{15} < 2.39816 \times 10^{-18}$$

$$\ldots$$

Strong Coupling Expansion from Dispersion Relation

$$\mathcal{O}(\lambda) = \int_0^{R^{-1}} dt \, \frac{\lambda}{1 + t\lambda} \nu(t) \longrightarrow$$

weak coupling expansion $|\lambda| < R$
given by moments of $\nu(t)$

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For the following discussion, I do not need to assume positivity of $\nu(t)$

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Mellin-Barnes representation
$$\boxed{\frac{\lambda}{1+t\lambda} = \int_{\delta+i\infty}^{\delta-i\infty} \frac{dj}{2\pi i} \frac{\pi}{\sin \pi j} \lambda^j t^{j-1} \qquad 0 < \mathrm{Re}\ \delta < 1}$$

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$\hat{\nu}(j)$

Mellin-transform

Analytically continue to $j \in \mathbb{C}$

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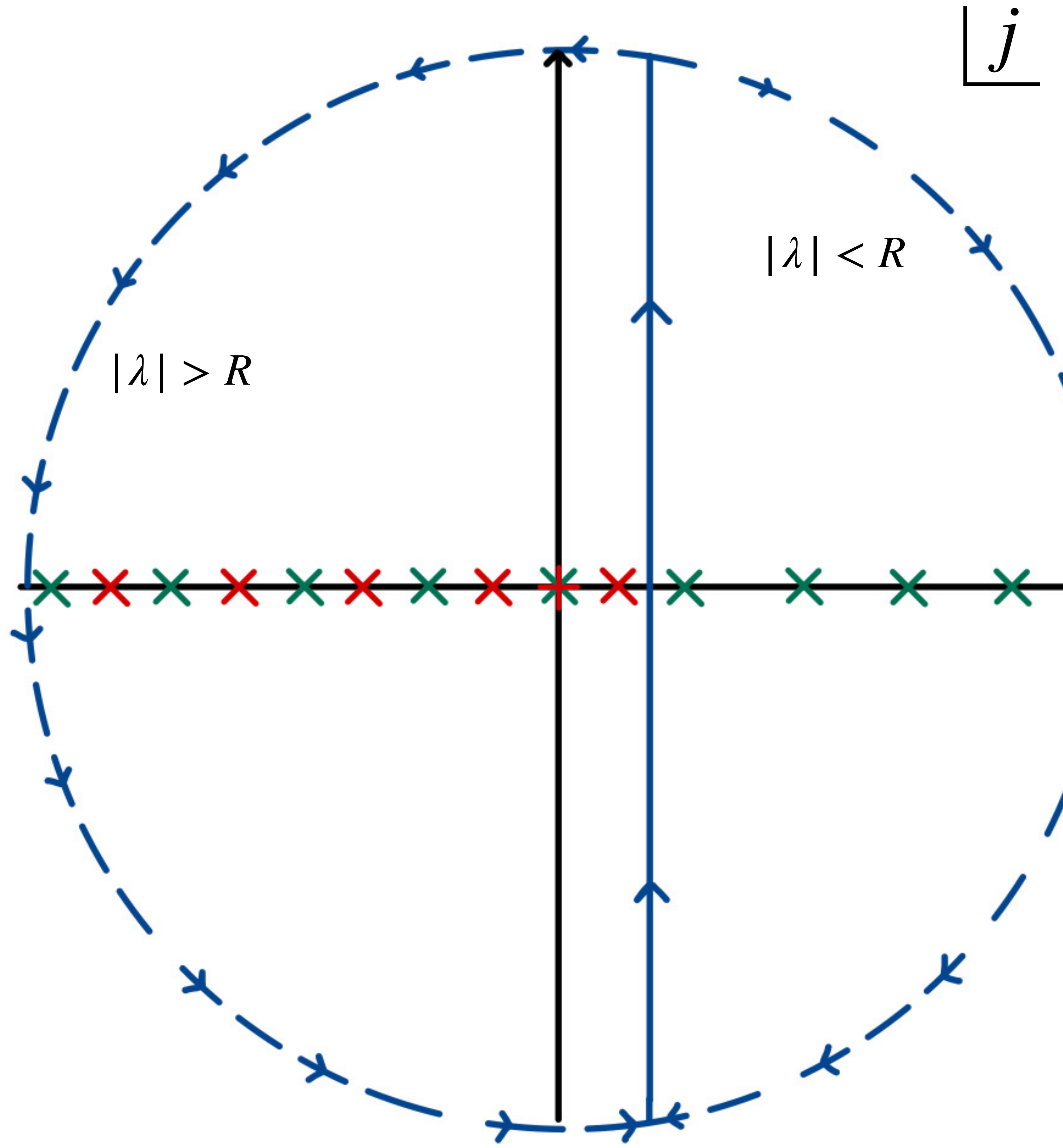

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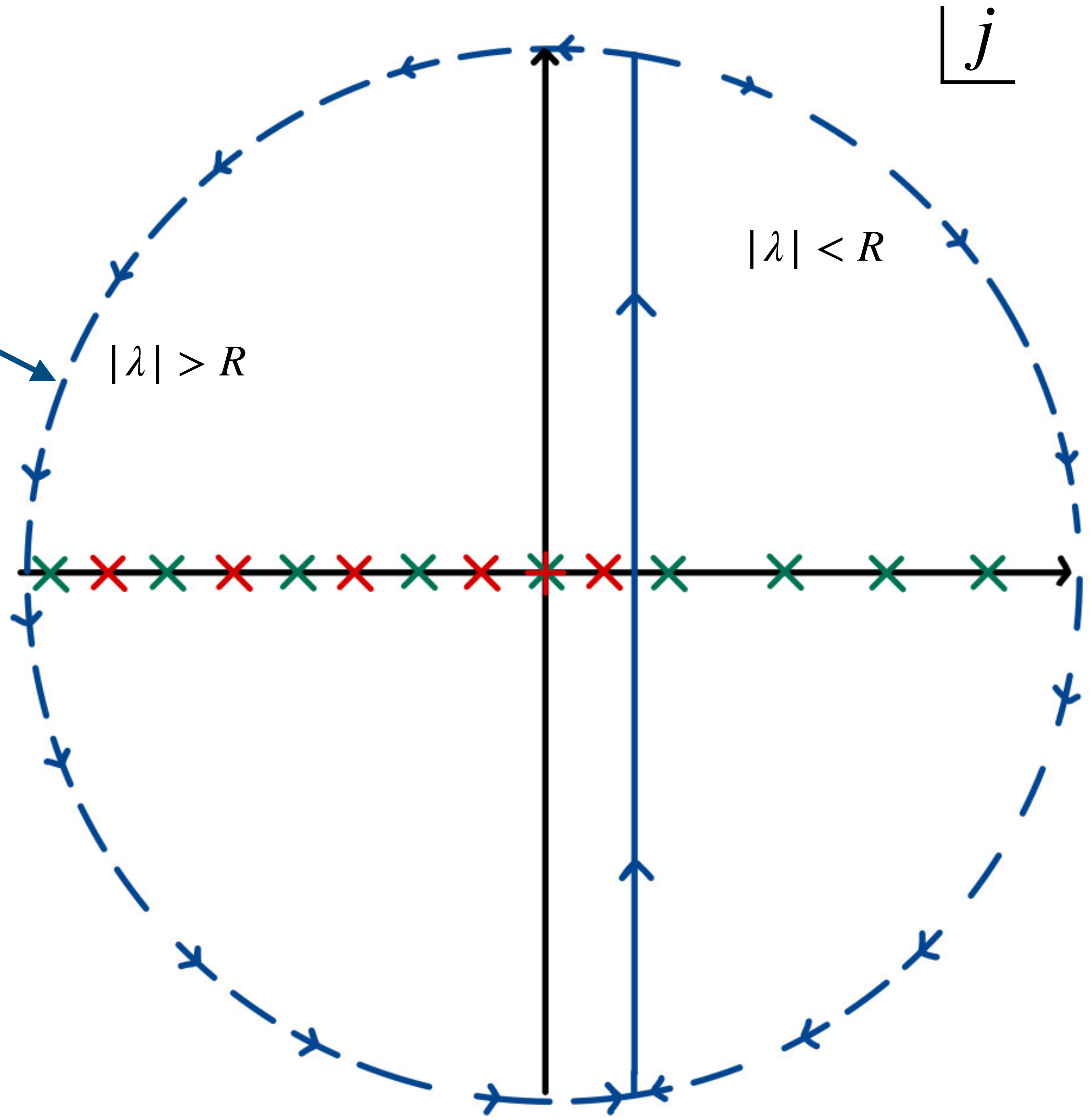
For $|\lambda| > R$ Close contour on the left side and pick up poles



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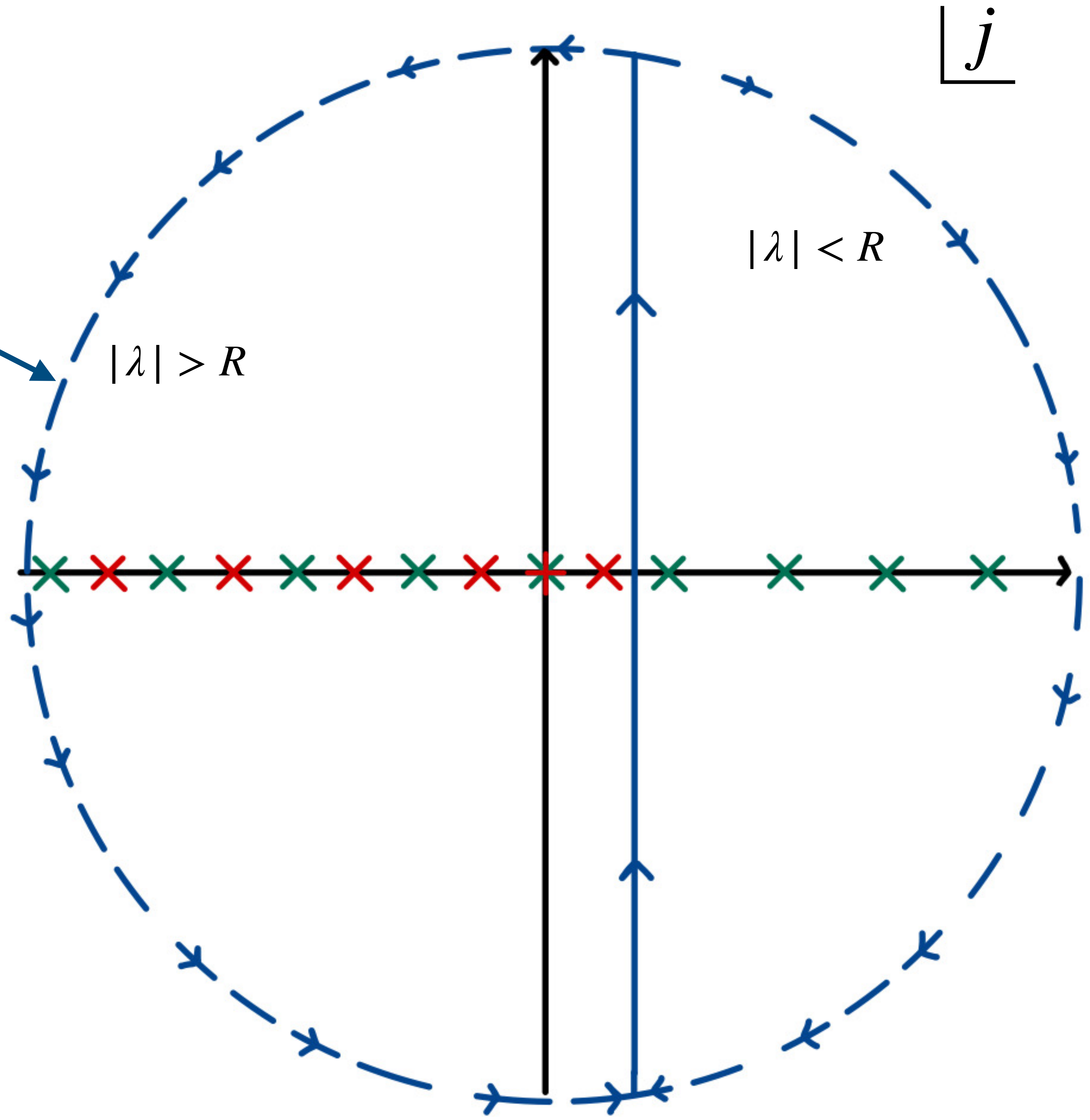
$$\Updownarrow$$

$\hat{\nu}(j)$ has poles at $j = \frac{1}{2}, 0, -\frac{1}{2}, -\frac{3}{2}, \dots$



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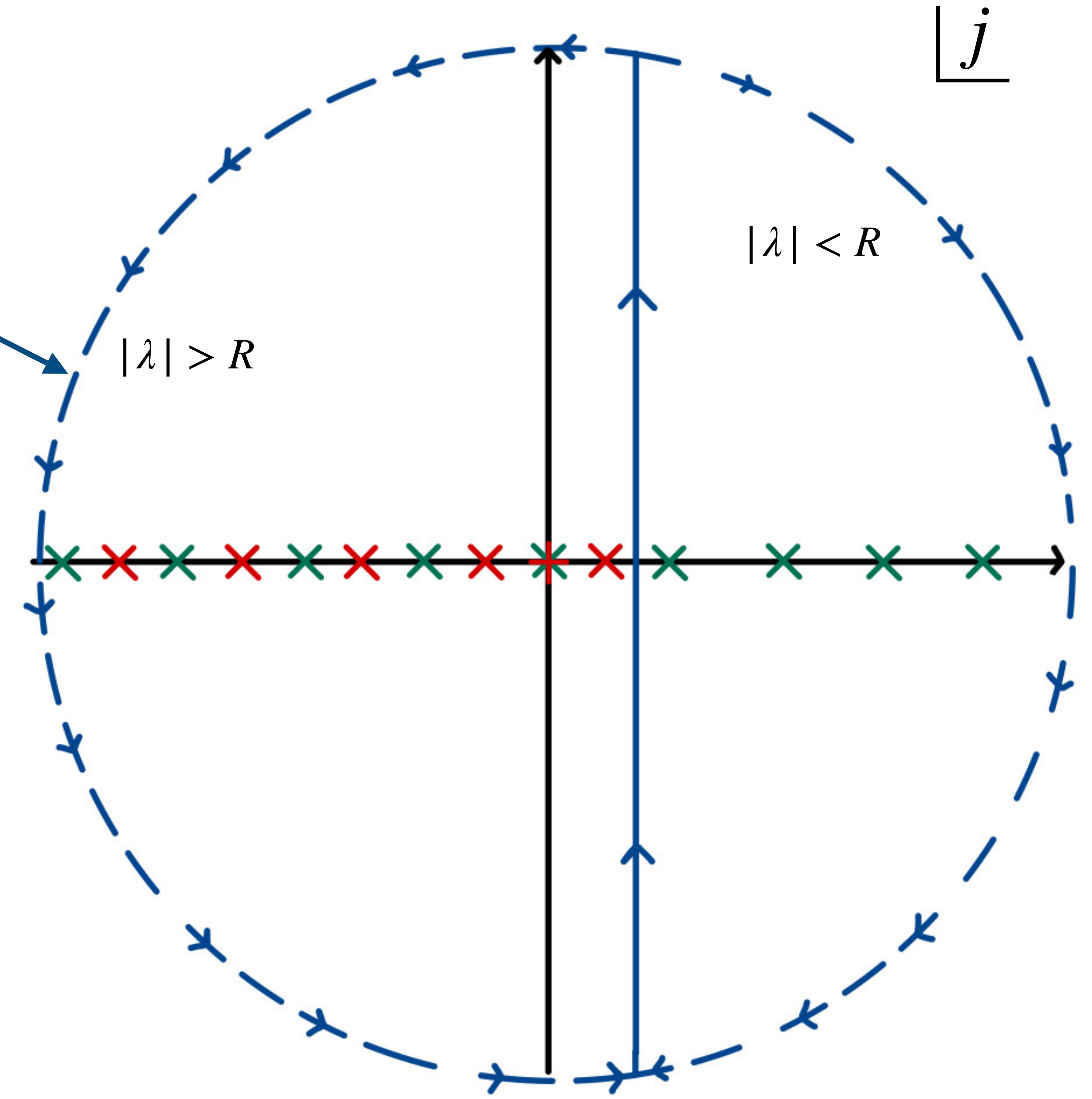
Exponentially suppressed terms $e^{-\sqrt{\lambda}}$ from saddle points



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$$\Gamma_{\text{oct}} = \int_{\delta-i\infty}^{\delta+i\infty} \frac{dj}{2\pi i} \frac{\pi}{\sin \pi j} \lambda^j \, \hat{\nu}(j) \qquad \hat{\nu}(n) = \int_0^{1/R} dt \, t^{n-1} \nu(t) = a_n$$

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$$\Gamma_{\text{oct}} = \log\cosh(2\pi g) = 2\pi^2 g^2 - \frac{4}{3}\pi^4 g^4 + \frac{64}{45}\pi^6 g^6 - \ldots$$

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Integrand has no other poles. There are no more perturbative contributions

Non-perturbative corrections from saddles

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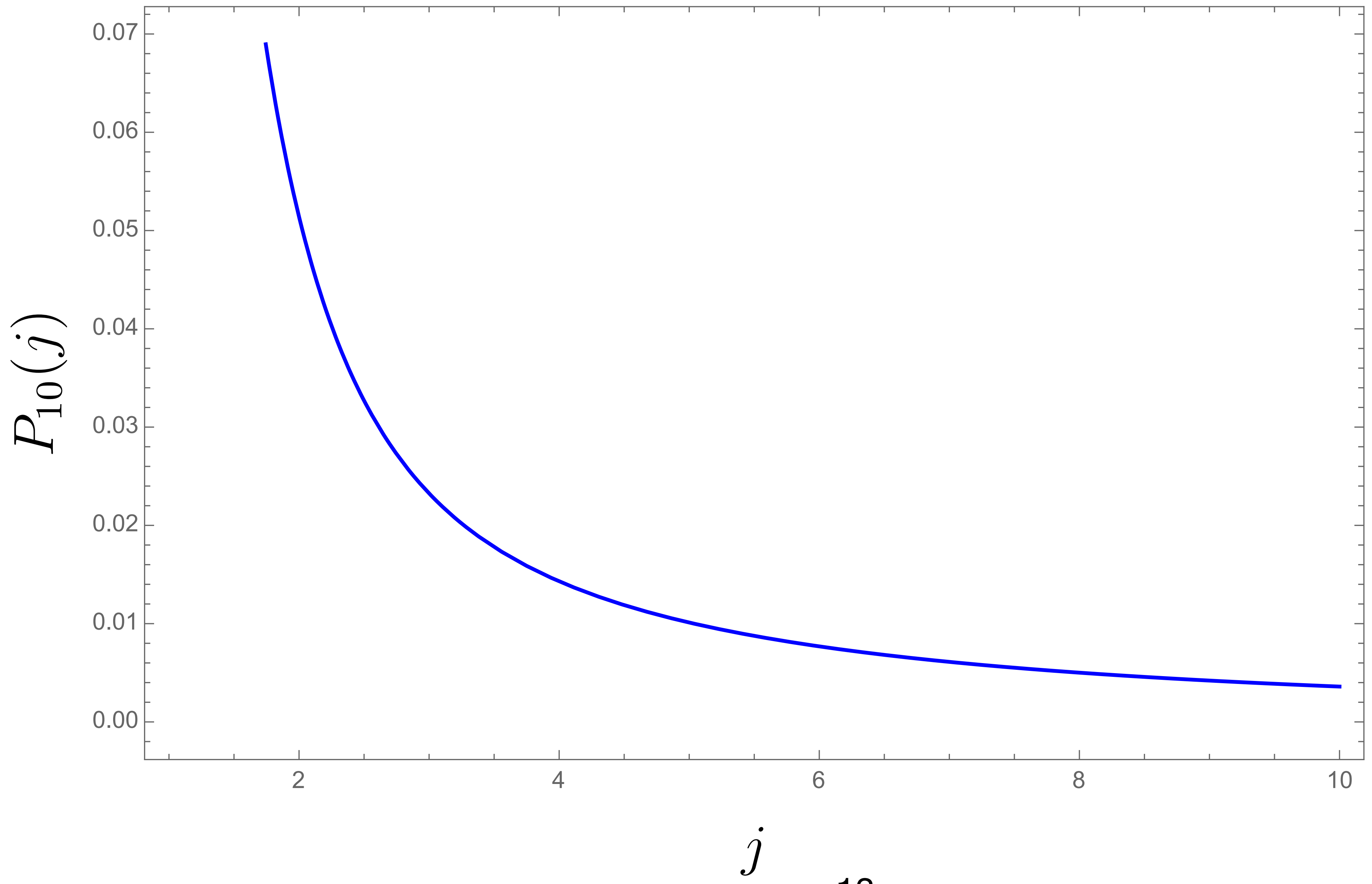

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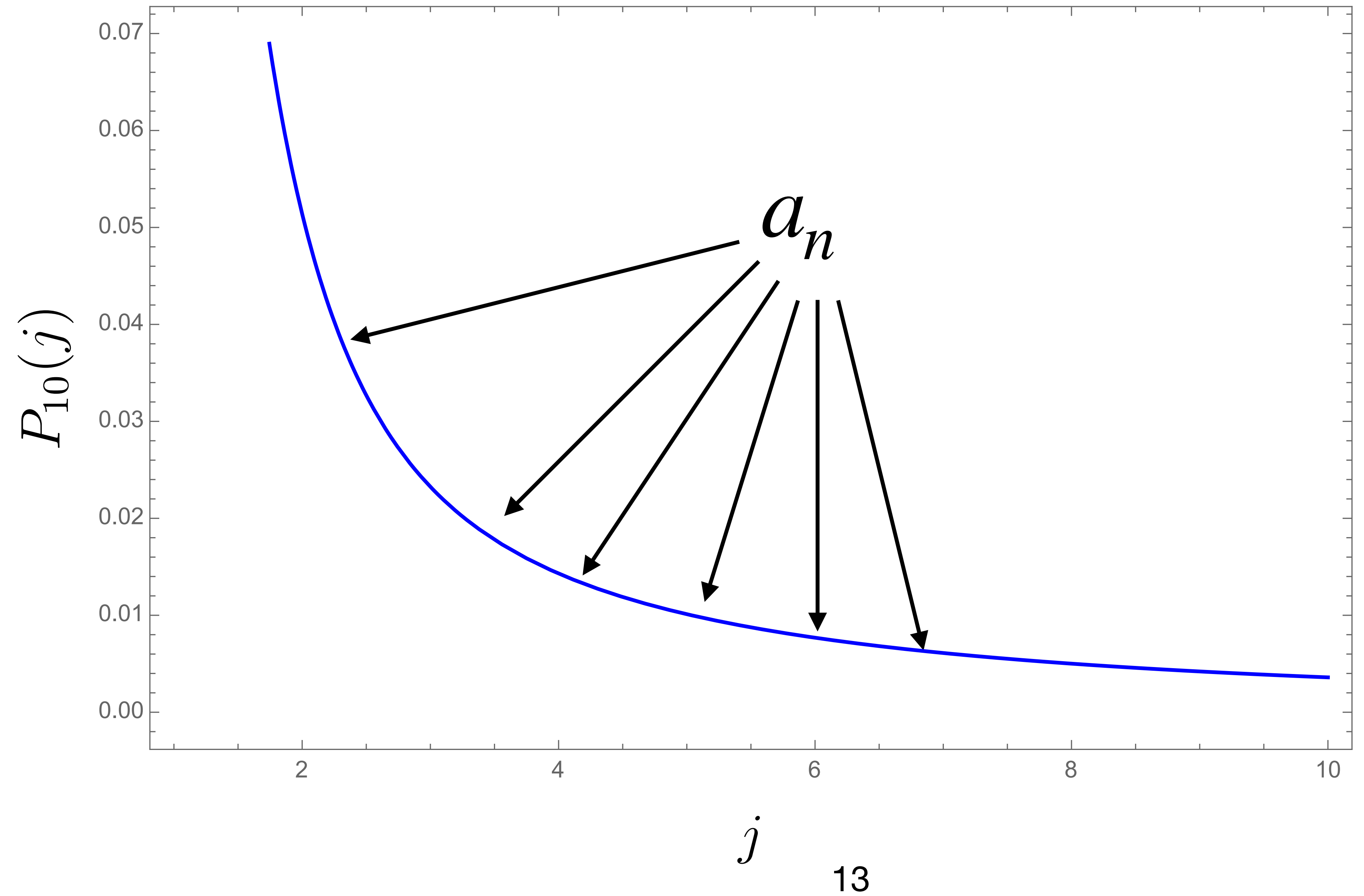

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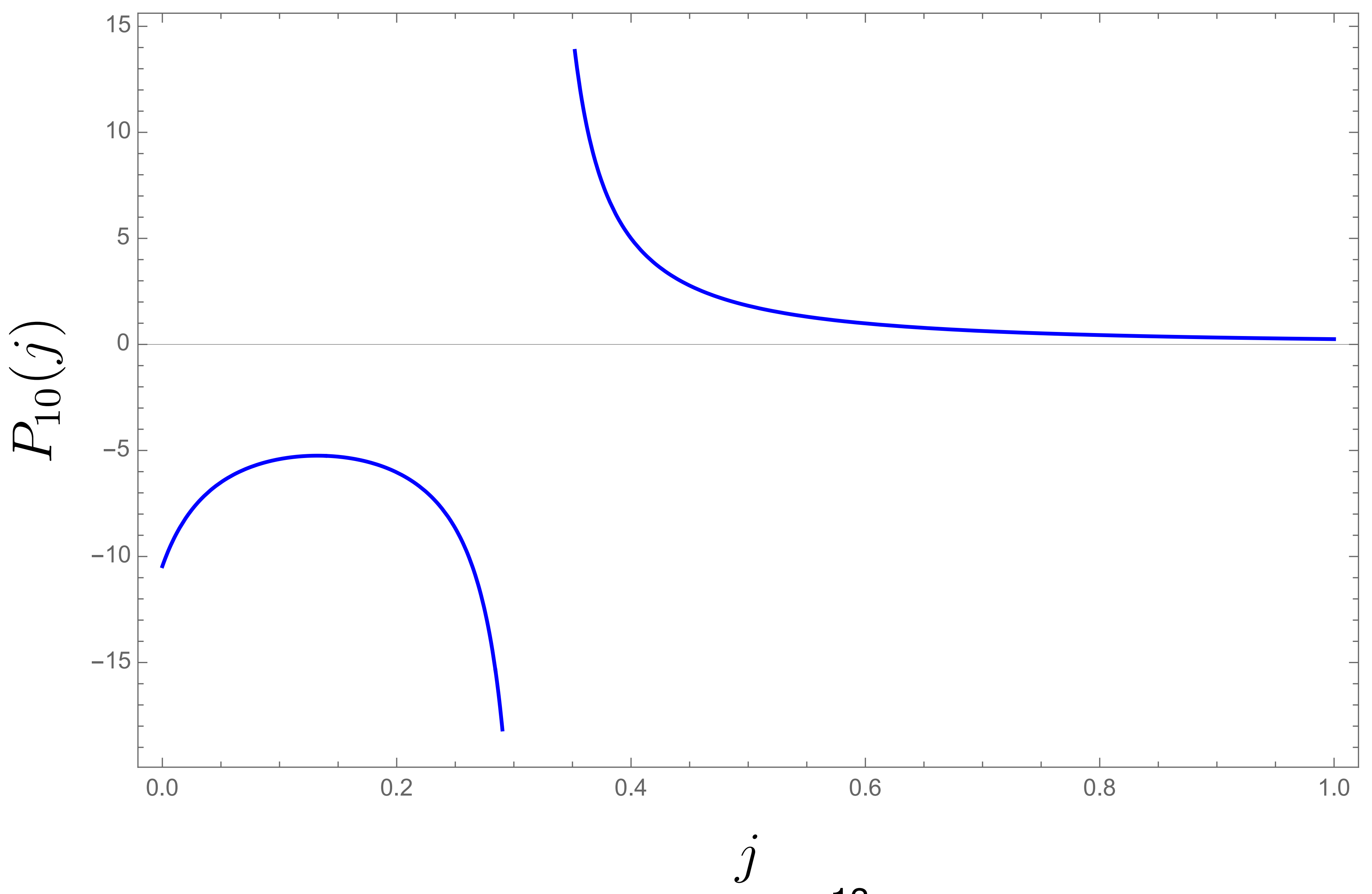

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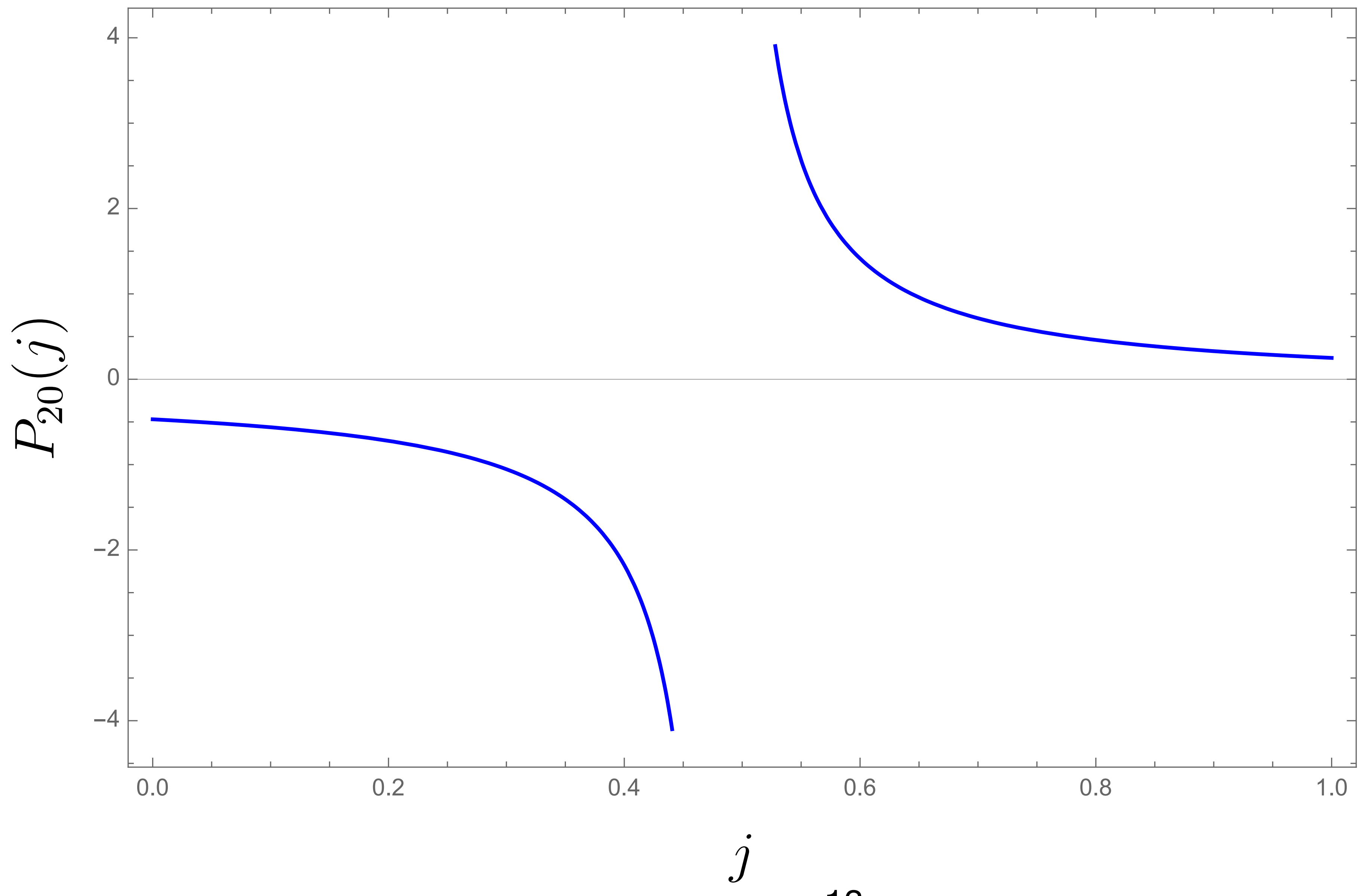

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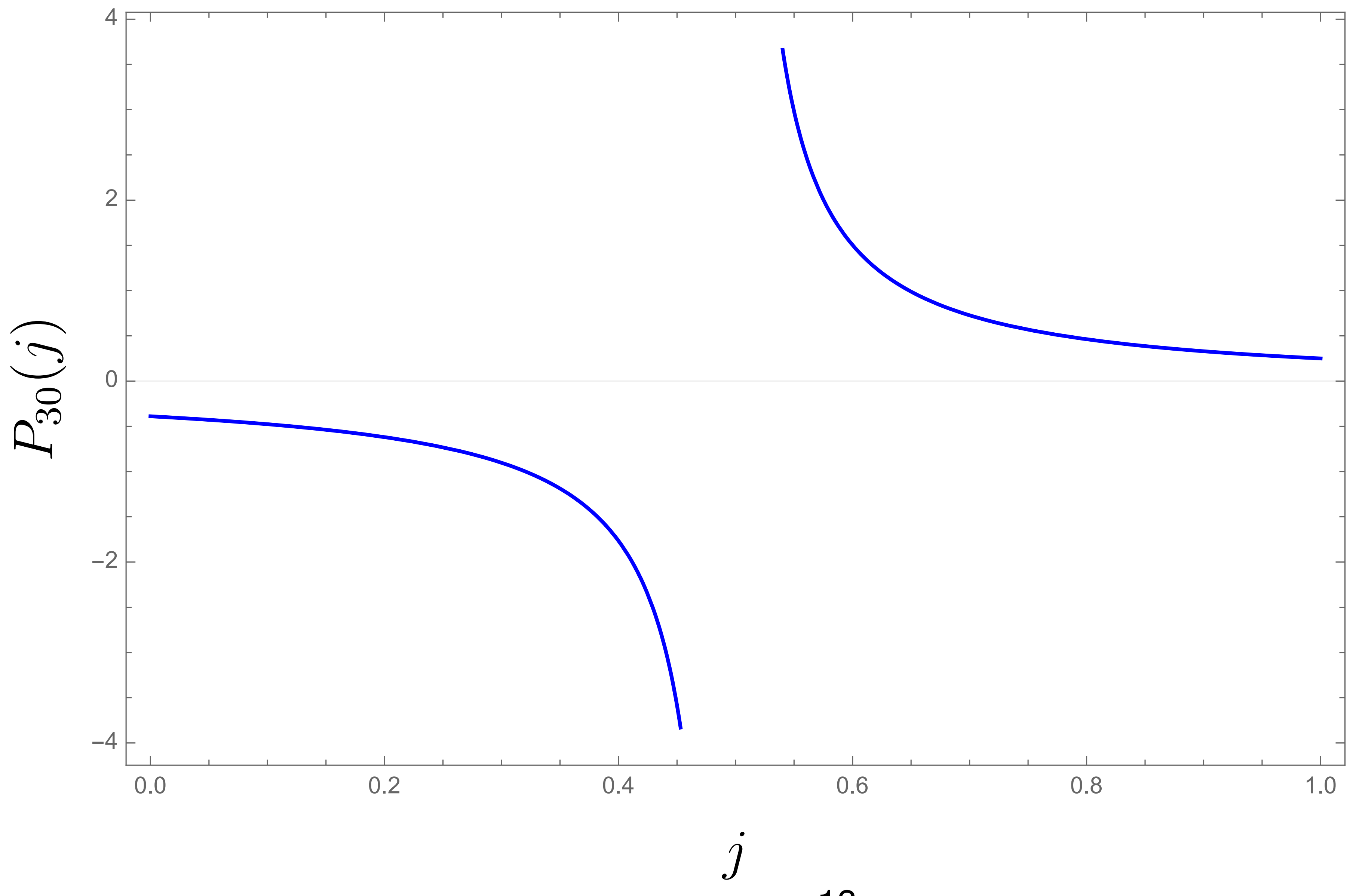

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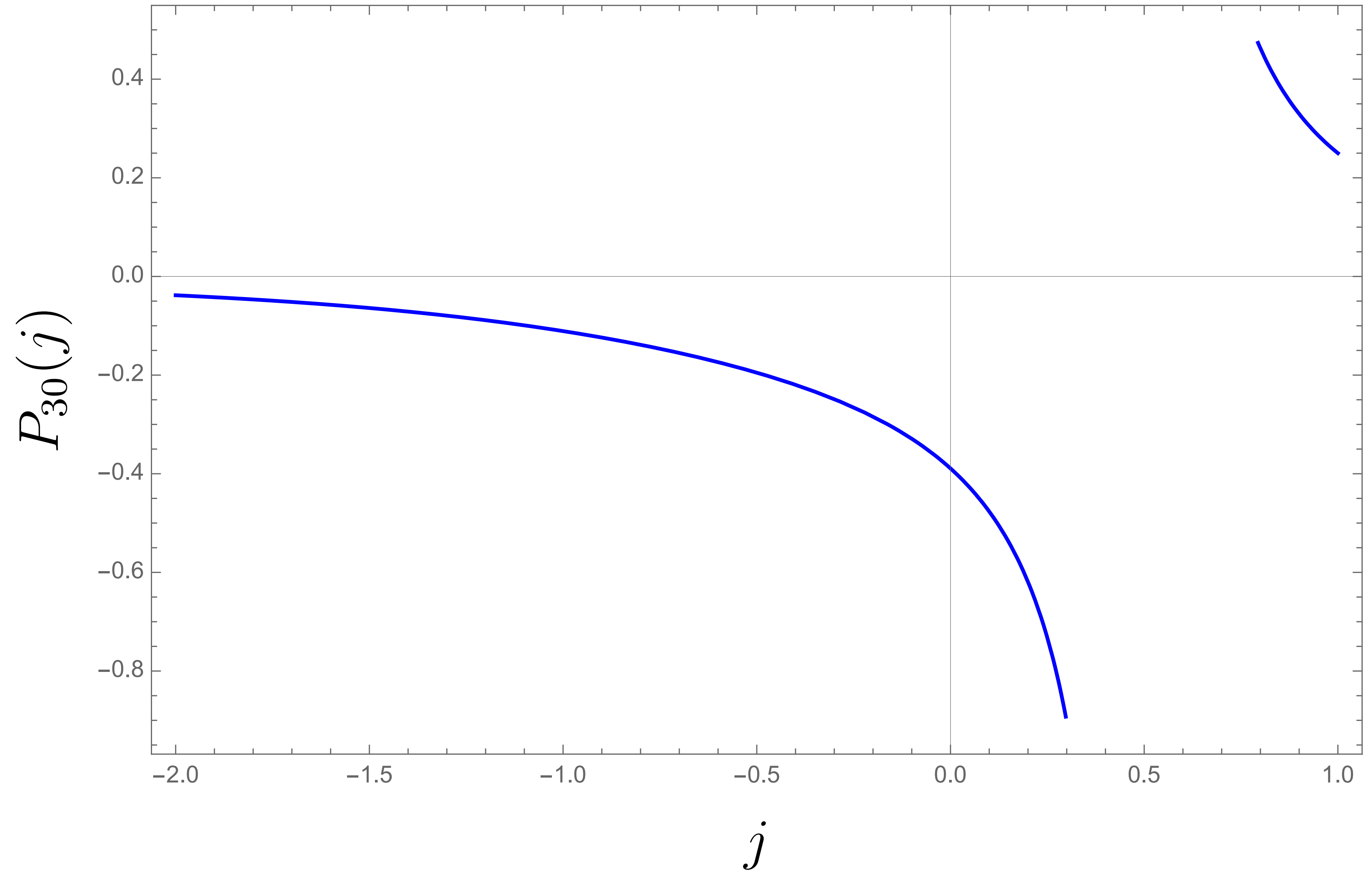

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There should be poles at $j = -\frac{1}{2}, -\frac{3}{2}, \ldots$

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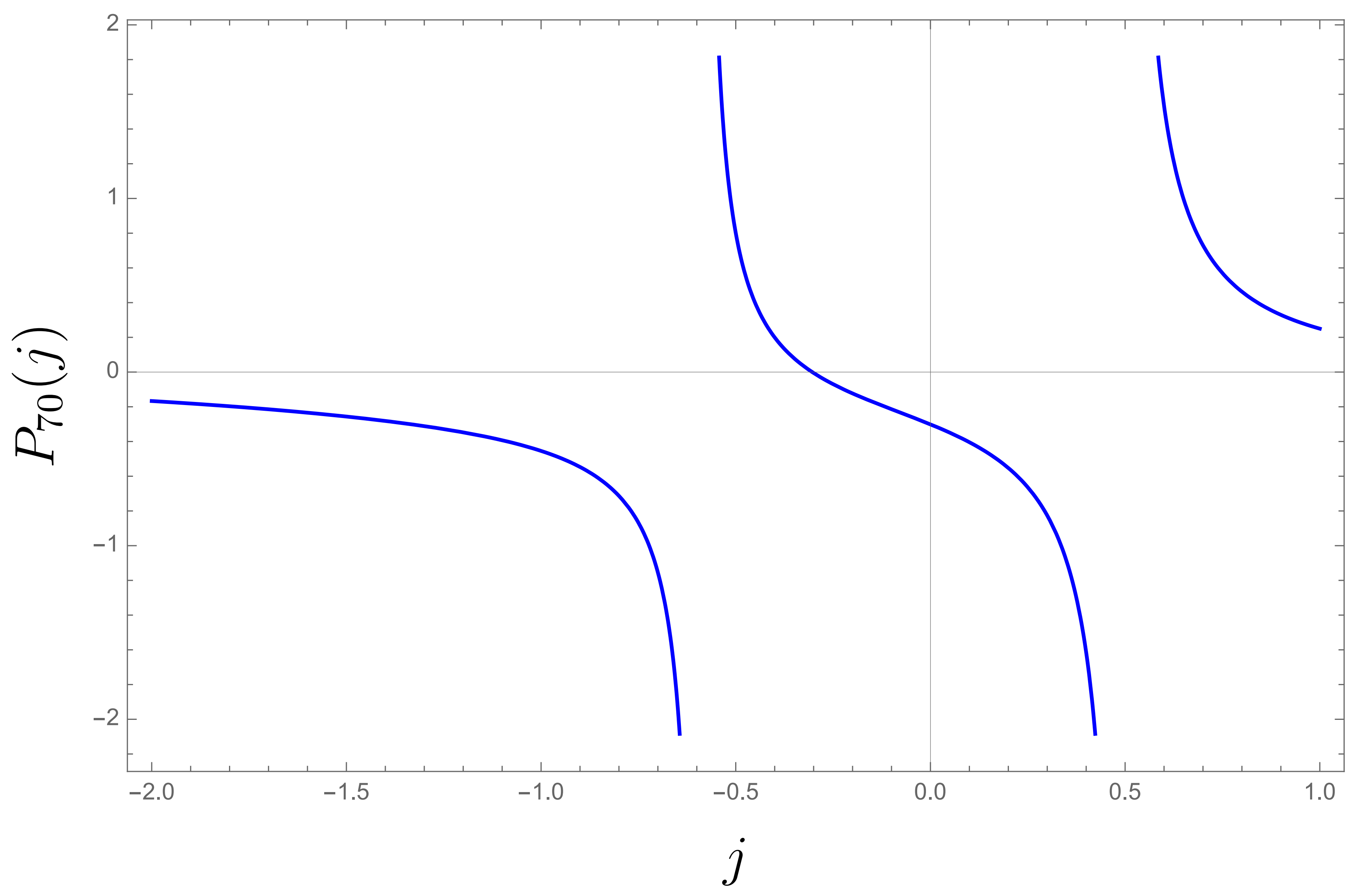

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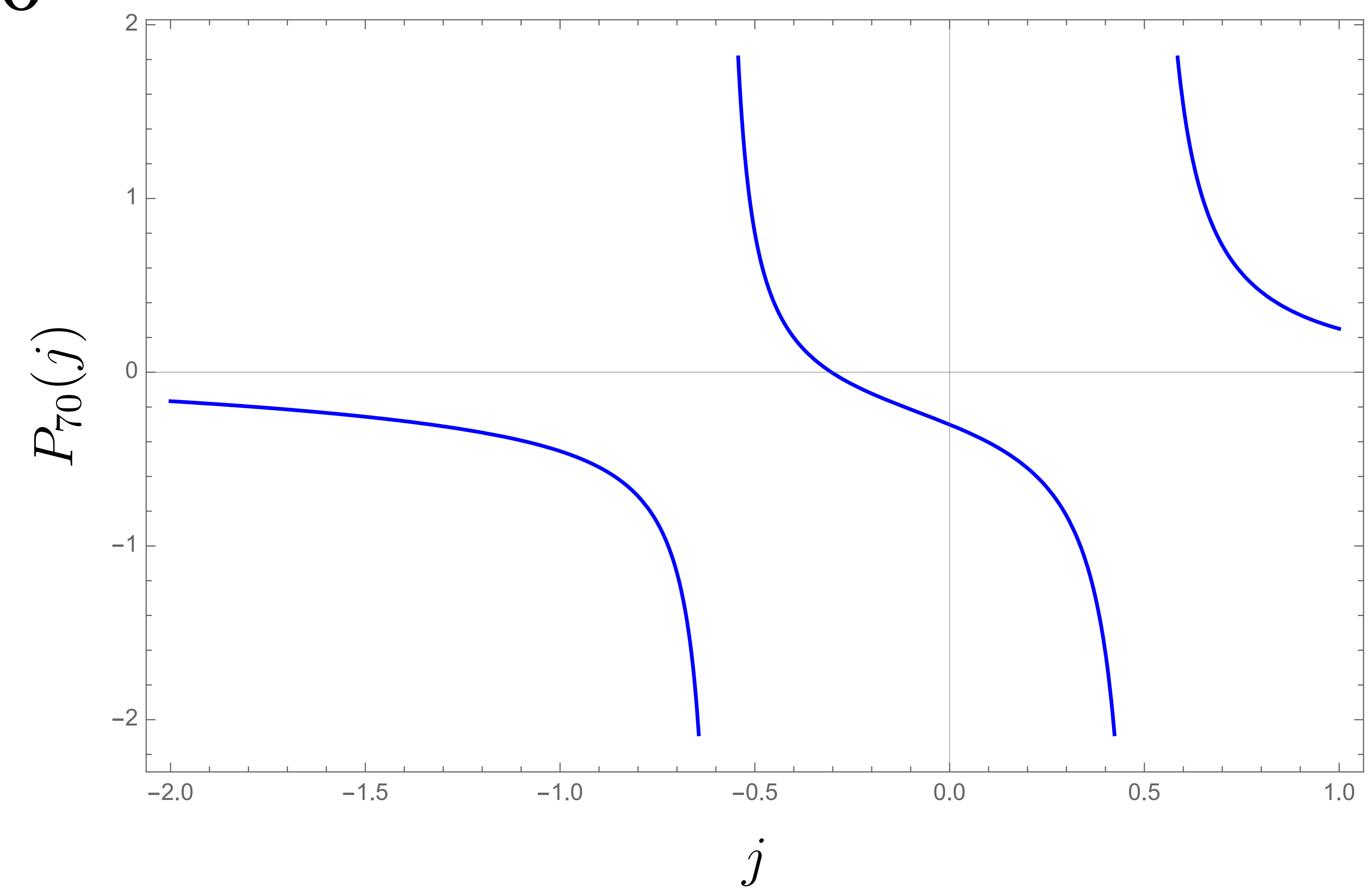

Plug $P_{70}(j)$ into Mellin-Barnes representation:

Padé: $\gamma_{\text{cusp}}(\lambda) \sim 0.496\,\lambda^{0.5008} - 0.301 - 0.310\,\lambda^{-0.590}$

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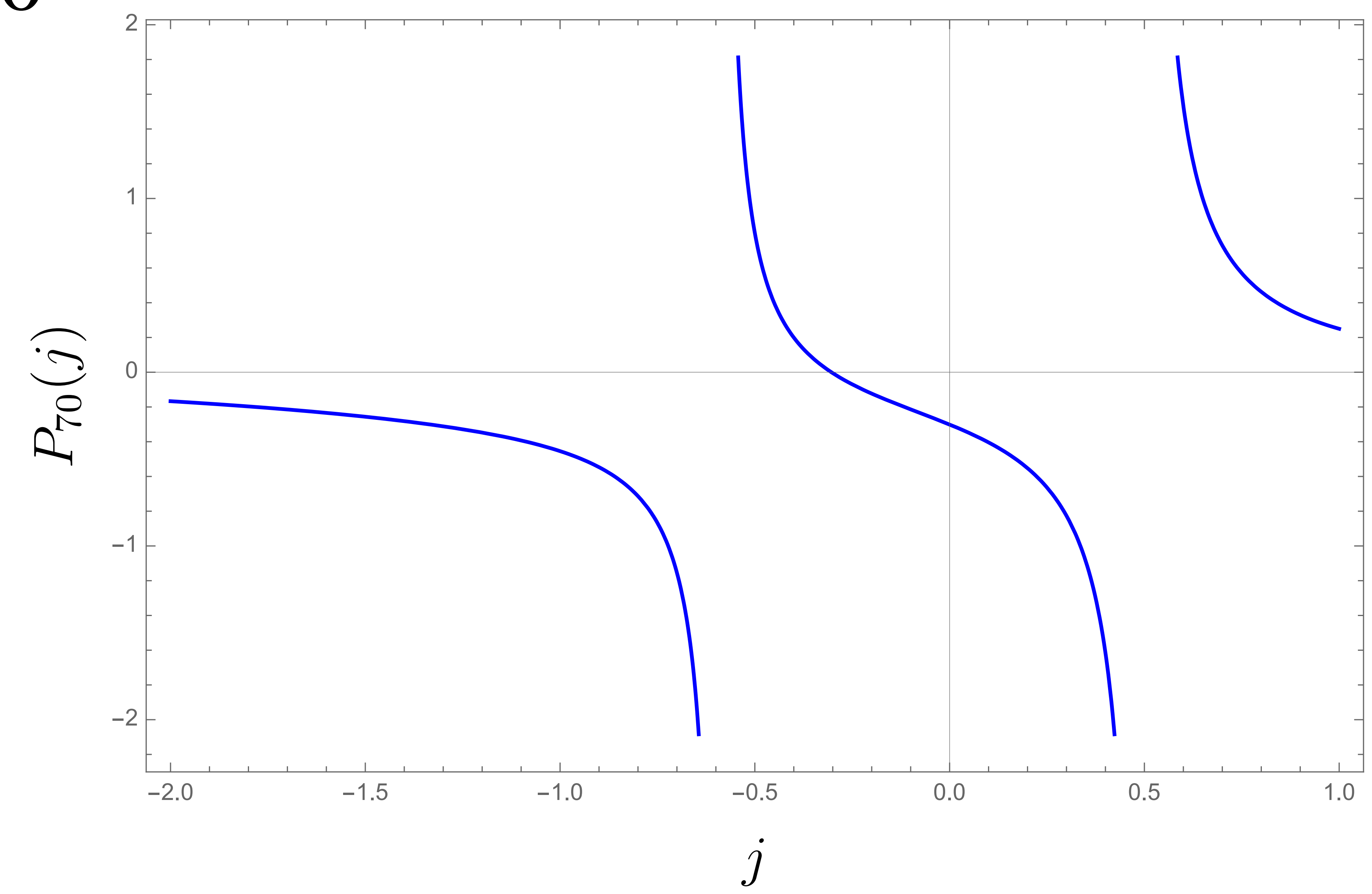

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- How can we make the analytical continuation of $\hat{\nu}(j)$ better?
- What is the connection between Stieltjes functions and conformal Padé approximants?

Thank you for your attention!